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Note

Numerical Computations on One-Dimensional Inverse Scattering Problems*

In this note we present an approximate method to detemine the index of refraction of a dielectric obstacle. For simplicity we treat one-dimensional models of electromagnetic scattering. The governing equations yield a second-order boundary value problem, in which the index of refraction appears as a functional parameter. The availability of reflection coefficients yields an additional initial condition. We approximate the index of refraction by a kth-order spline which can be written as a linear combination of B-splines. For $N/2$ distinct reflection coefficients, the resulting $N/2$ initial value problems yield a system of N nonlinear equations in N unknowns which are the coefficients of the B-splines.

1. Introduction

In this note we treat a class of inverse scattering problems in one dimension. We seek a function $n(x)$, which we refer to as the index of refraction, such that the solution of the boundary value problem

$$\begin{aligned} &u''(x) + n^2(x)\omega^2 u(x) = 0, \qquad x \in (0, 1) \\ &(u' + in_0\omega u)(0) = 2in_0\omega \\ &(u' - in_0\omega u)(1) = 0 \end{aligned} \tag{B}$$

satisfies $u(0) = 1 + R(\omega)$ and $u'(0) = in_0\omega(1 - R(\omega))$, where $R(\omega)$ is called the reflection coefficient. In practice $R(\omega)$ can be measured for any desired value of ω. The problem can be thought of as arising from the refraction of an incident wave of the form $e^{in_0\omega x}$ by a dielectric obstacle whose index of refraction is unknown. See [1] for a complete physical derivation of the boundary value problem (B).

This problem has also been studied by Hagin [2] and Gray and Hagin [3]. A similar problem has been studied by Schaubert and Mittra [4] and Tsien and Chen [5].

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2. Solution Procedure

Our solution procedure consists of approximating $n(x)$ with a kth-order spline $\bar{n}(x)$. This choice of approximation is motivated by the need to solve the initial value problem

$$\begin{aligned} u'' + \omega^2 \bar{n}^2 u &= 0, \qquad x \in (0, 1) \\ u(0) &= 1 + R(\omega) \\ u'(0) &= in_0 \omega (1 - R(\omega)) \end{aligned} \tag{P}$$

quickly and accurately.

We begin with a brief description of the spline spaces $S_k(\mathbf{x})$. Let $\mathbf{x} = \{x_i\}_{i=1}^{N+1}$ be a partition of $[0, 1]$, and let $S_k(\mathbf{x})$ denote the space of kth-order splines with knots at each x_i. Thus, $s \in S_k(\mathbf{x})$ implies that in each interval $[x_i, x_{i+1}]$, s is a polynomial of degree at most $k-1$ and $s \in C^{(k-2)}[0, 1]$. Let $\{B_{i,k}\}_{i=1}^{N+k-1}$ denote the B-spline basis for $S_k(\mathbf{x})$. The general properties of B-splines are well known and can be found in [6], for example. A property that we find convenient for our numerical scheme is that $B_{i,k}(x) = 0$ if $x \notin [x_{i-k+1}, x_{i+1}]$. As a consequence, if $f(x) = \sum_{i=1}^{\bar{N}} \alpha_i B_{i,k}(x)$ and if $x \in [x_m, x_{m+1}]$, where $\bar{N} = N + k - 1$, then $f(x)$ is simply given by

$$f(x) = \sum_{i=m}^{m+k-1} \alpha_i B_{i,k}(x). \tag{2.1}$$

The approximation of n proceeds as follows: For given N and k we seek an approximation $\bar{n}(x) = \sum_{i=1}^{\bar{N}} \lambda_i B_{i,k}(x)$ such that, given the distinct pairs $(\omega_j, R(\omega_j))$, $j = 1,..., l$, where $l = \lfloor \bar{N}/2 \rfloor$, the solutions of the l initial value problems (P) satisfy $(u' - i\omega_j u)(1) = 0$. This is equivalent to solving

$$\mathbf{F}(\boldsymbol{\lambda}) = \mathbf{0} \tag{2.2}$$

where $\boldsymbol{\lambda} = (\lambda_1,..., \lambda_N)^T$ and $\mathbf{F}(\boldsymbol{\lambda}) = (f_1(\boldsymbol{\lambda}),..., f_N(\boldsymbol{\lambda}))^T$ with $f_{2j} = \text{Real}\ ((u' - i\omega_j u)(1))$ and $f_{2j-1} = \text{Imag}((u' - i\omega_j u)(1))$, $j = 1, 2,..., l$.

In order to solve (2.2) one must solve (P) many times. In general this requires the numerical solution of these problems which leads to a large cost in computer time. Our choice of approximation reduces this time to a great extent. For example, when $k = 1$, $\bar{n}$ is piecewise constant, thus one can obtain the solution of (P) in closed form, and when $k \geqslant 2$ one can easily obtain symbolic Taylor series expansions of u. The Taylor expansions are computed as follows: For $x \in [x_m, x_{m+1}]$, $\bar{n}(x)$ is a polynomial of degree at most $k-1$ and is given by $\bar{n}(x) = \sum_{i=m}^{m+k-1} \lambda_i B_{i,k}(x)$. This follows from (2.1). Let $S_m(x)$ and $H_m(x)$ denote two linearly independent Taylor series solutions, expanded about x_m, of $(\text{P})_1$ that satisfy $S_m(x_m) = H'_m(x_m) = 1$ and $S'_m(x_m) = H_m(x_m) = 0$. Let $a_1 = 1 + R(\omega)$ and $b_1 = in_0 \omega (1 - R(\omega))$, then for $x \in [x_m, x_{m+1}]$ the

solution of (P) is given by $u(x) = a_m S_m(x) + b_m H_m(x)$, where $a_m = a_{m-1} S_{m-1}(x_m) + b_{m-1} H_{m-1}(x_m)$ and $b_m = a_{m-1} S'_{m-1}(x_m) + b_{m-1} H'_{m-1}(x_m)$ for $m = 2, 3, \ldots, N + k$. For comparative purposes the solution of (2.2) was obtained by solving (P) with a fourth-order, variable step size Runge and Kutta integrator. This resulted in a fivefold increase in computer time.

It is known [6] that if $f \in C^{(j)}[0, 1]$ for $j = 0, 1, \ldots, k-1$, then $\inf_{s \in S_k(\mathbf{x})} \| f - s \|_\infty \leqslant C_{k,j} h^j w(f^{(j)}; h)$, where $h = \max_{1 \leqslant i \leqslant N}(x_{i+1} - x_i)$, $C_{k,j}$ is a constant which depends only on k and j and

$$w(f^{(j)}; h) := \sup\{|f^{(j)}(x) - f^{(j)}(y)| : x, y \in [0, 1], |x - y| \leqslant h\}$$

is the modulus of continuity of $f^{(j)}$ at h. It is also known that the above estimate cannot be improved. Thus, if $n \in C^{(k)}[0, 1]$, then $w(n^{(k-1)}; h) \leqslant h \| n^{(k)} \|_\infty$ and the best that we can hope for is $\| n - \bar{n} \|_\infty = O(h^k)$. As will be seen, in several numerical examples tested the optimal convergence rate was attained.

3. Numerical Results

In this section we present some numerical examples for $k = 1, 2$ and 3 with knot sequence $\{(i-1)/N\}_{i=1}^{N+1}$. In all examples the Levenberg–Marquardt algorithm [7] was used to solve (2.2). The $R(\omega_j)$'s were computed by inputting the exact solution $n(x)$ into (B). All computations were done on the Cyber-173.

To solve a large nonlinear system of equations it is usually necessary to have a good initial approximation to the solution. For fixed N and k our solution was built up according to the following algorithm:

(i) Select a sequence of integers $\mathbf{m} = \{m_i\}_{i=1}^{M}$ with $m_1 \geqslant k$, $m_M = \bar{N}$ and $m_i < m_{i+1}$.

(ii) Compute $\bar{n}_{m_1}$ by solving (2.2) with an initial guess $\lambda_1 = \cdots = \lambda_{m_1} = 1$.

(iii) For $j = 1, \ldots, M-1$ compute $\bar{n}_{m_{j+1}}$ by solving (2.2) with the initial guess chosen so that $\bar{n}_{m_{j+1}}$ interpolates $\bar{n}_{m_j}$ at the m_{j+1} distinct points $\{y_i\}_{i=1}^{m_{j+1}}$. This gives a linear system for the initial λ_i's that is invertible if and only if

$$x_i < y_i < x_{i+k}, \qquad i = 1, \ldots, m_{j+1} \quad [6].$$

We found that adequate results were obtained by setting $\mathbf{m} = \{2^i\}_{i=1}^{M}$ for $k = 1$, and $\mathbf{m} = \{4i\}_{i=1}^{M}$ for $k = 2$ and 3.

The maximum attainable $\bar{N}$ is limited by considerations of computer time and accuracy. In the examples tested we found that satisfactory results were obtained if $\bar{N} = 32$ for $k = 1$ and $\bar{N} = 8$ or 12 for $k = 2$ and 3.

EXAMPLE 1 (FIGS. 1, 5, 9).

$$n(x) = 1 + x^2.$$

TABLE I

k	1	2	3
$\bar{N}$	32	12	12
$\|n - \bar{n}\|_\infty$	0.033	0.0033	1×10^{-7}
$\|n - \bar{n}\|_2$	0.014	0.0011	0
Computational time (CPU sec)	74	22	24
Estimated convergence rate	1.0	2.1	—

EXAMPLE 2 (FIGS. 2, 6, 10).

$$n(x) = \begin{cases} 1, & x \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1] \\ 1 + \sin^2 2\pi(x - \frac{1}{4}), & x \in [\frac{1}{4}, \frac{3}{4}]. \end{cases}$$

TABLE II

k	1	2	3
$\bar{N}$	32	12	12
$\|n - \bar{n}\|_\infty$	0.12	0.071	0.065
$\|n - \bar{n}\|_2$	0.032	0.18	5.5×10^{-4}
Computational time (CPU sec)	120	41	33
Estimated convergence rate	1.1	0.96	—

EXAMPLE 3 (FIGS. 3, 7, 11).

$$n(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}) \\ 2, & x \in [\frac{1}{2}, 1]. \end{cases}$$

TABLE III

k	1	2	3
$\bar{N}$	16	8	8
$\|n - \bar{n}\|_\infty$	0.49	0.48	0.46
$\|n - \bar{n}\|_2$	0.17	0.10	0.013
Computational time (CPU sec)	27	12	7.0
Estimated convergence rate	—	—	—

EXAMPLE 4 (FIGS. 4, 8, 12).

$$n(x) = \begin{cases} 1, & x \in [0, \frac{1}{4}) \\ \frac{1}{2} + 2x, & x \in [\frac{1}{4}, \frac{3}{4}] \\ 2, & x \in (\frac{3}{4}, 1]. \end{cases}$$

TABLE IV

k	1	2	3
$\bar{N}$	32	12	12
$\|n - \bar{n}\|_\infty$	0.038	0.024	0.023
$\|n - \bar{n}\|_2$	0.015	0.0074	4.4×10^{-5}
Computational time (CPU sec)	168	46	24
Estimated convergence rate	1.1	1.1	—

In practice the measurement of the reflection coefficients is subject to experimental error. To simulate this situation we introduced Gaussian-type noise in the values of $R(\omega)$ in the form $R(\omega) + \varepsilon a/|R(\omega)|$, where a is Gaussian random number with mean $=0$ and standard derivation $=1$ and ε is an "amplitude" factor to be adjusted. As the following tables indicate stable results were obtained when ε was less than 10^{-3}.

EXAMPLE 5.

$$n(x) = 1 + x^2, \qquad k = 2, \quad \bar{N} = 8.$$

TABLE V

Comp. time	$\varepsilon/\|R(\omega)\|$	$\|n - \bar{n}\|_2$	$\|n - \bar{n}\|_\infty$
6.0	0	0.0019	0.0054
7.9	10^{-5}	0.0093	0.026
12	10^{-4}	0.070	0.19
25	10^{-3}	0.34	1.0

EXAMPLE 6.

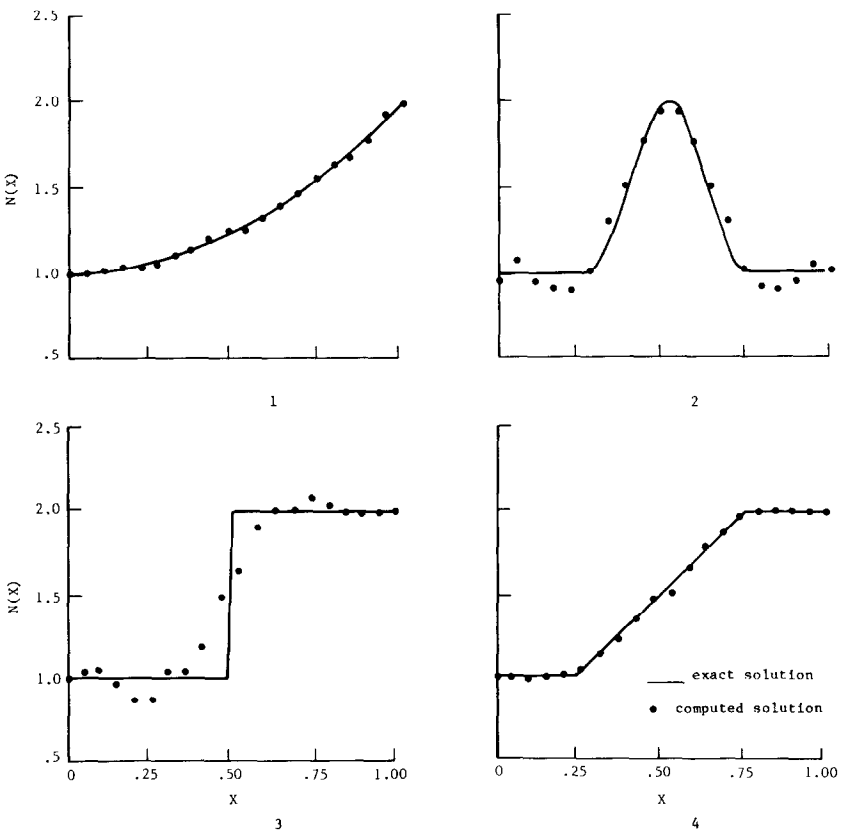
$$n(x) = \begin{cases} 1, & x \in [0, \frac{1}{4}) \\ \frac{1}{2} + 2x, & x \in [\frac{1}{4}, \frac{3}{4}], \\ 2, & x \in [\frac{3}{4}, 1]. \end{cases} \qquad k = 2, \bar{N} = 8$$

TABLE VI

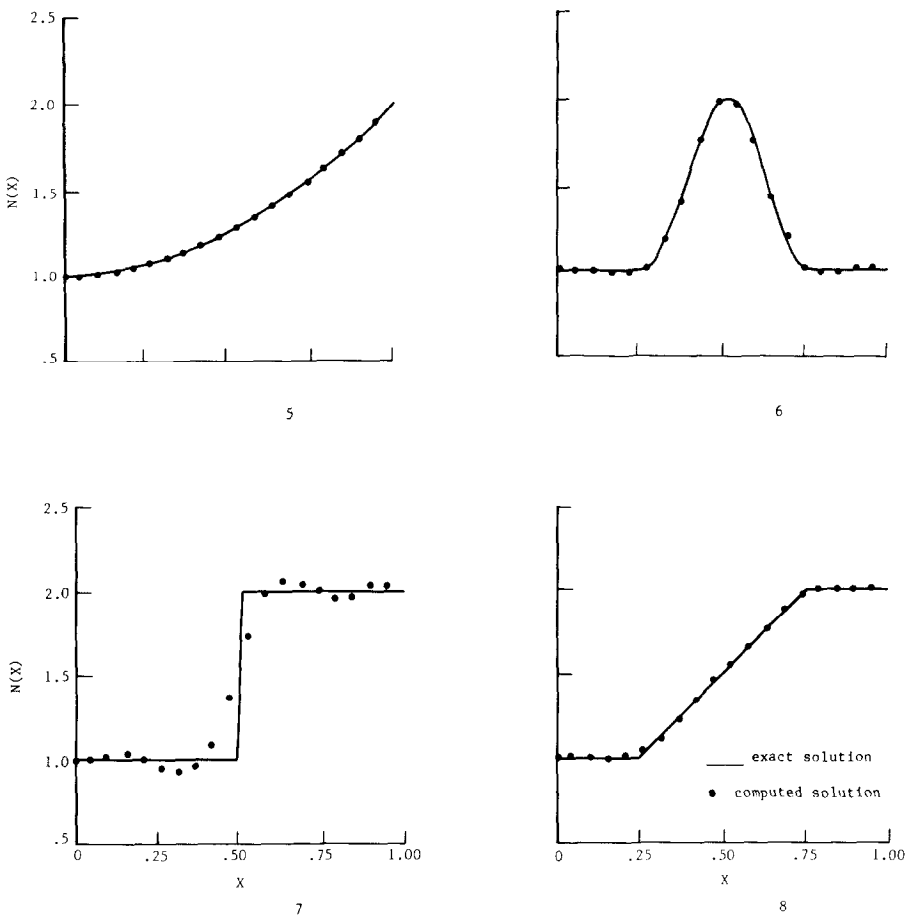
Comp. time	$\varepsilon/\lvert R(\omega)\rvert$	$\lVert n-\bar{n}\rVert_2$	$\lVert n-\bar{n}\rVert_\infty$
14	0	0.22	0.061
19	10^{-5}	0.012	0.031
22	10^{-4}	0.081	0.20
34	10^{-3}	0.38	1.1

4. Discussion

In all noiseless examples with $k = 1$ and 2, except for Example 3, the optimal convergence rate was attained.



Figs. 1–4. $k = 1$.

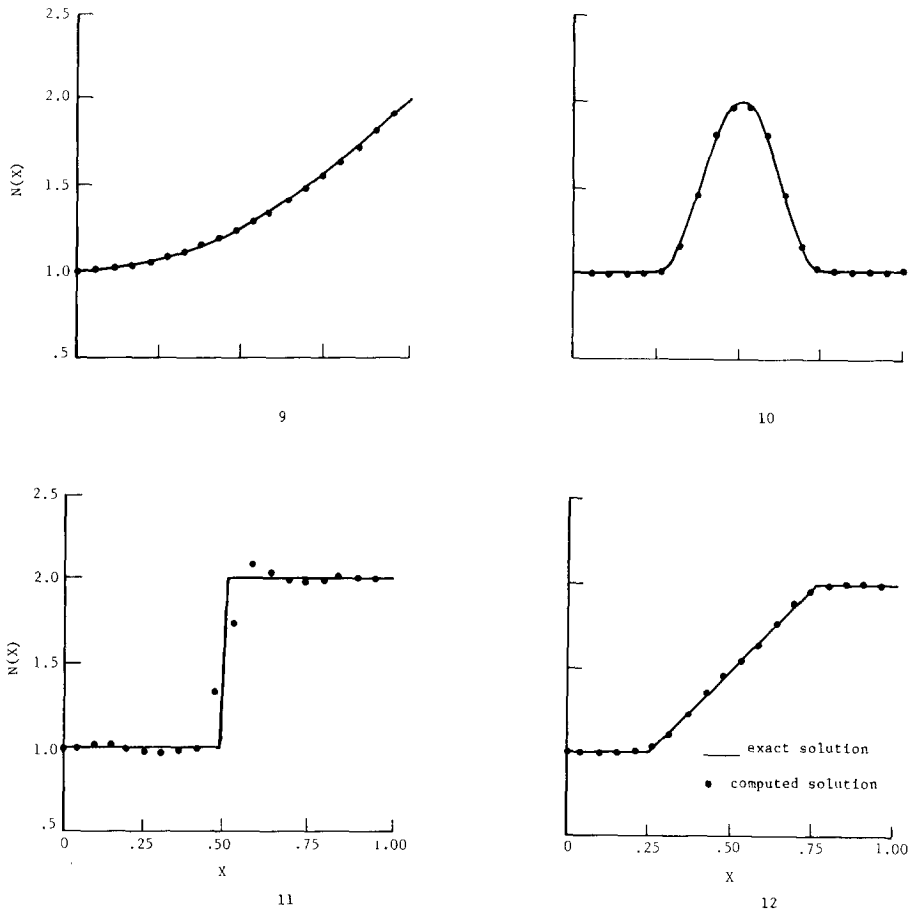


FIGS. 5–8. $k = 2$.

It is obvious from the results listed in Tables I–IV that if a desired accuracy is specified, then the minimum computational time is achieved by increasing the order of the splines rather than increasing the number of unknowns for a fixed order.

The results in Tables V and VI show that our method is relatively stable with respect to experimental error which is of fundamental importance in most applications.

The numerical approaches used in [2] and [3] require knowledge of the reflection coefficient for values of frequency equal to $j\pi/2$ for $j = 1,..., J$. Our method has no such restriction. In addition, accuracy comparable to that obtained by the above is achieved with far fewer reflection coefficient observations (typically, 2–4 times less).



FIGS. 9–12. $k = 3$.

The pulse spectrum technique (PST) used in [5] also avoids the above difficulties. However, due to the "analytic" solution of the initial value problem (P), and the overall simplicity of our method, it seems to be an attractive alternate to the PST.

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